

On the Szegő–Asymptotics for Doubly–Dispersive Gaussian Channels

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Abstract—We consider the time–continuous doubly–dispersive channel with additive Gaussian noise and establish a capacity formula for the case where the channel correlation operator is represented by a symbol which is periodic in time and fulfills some further integrability and smoothness conditions. The key to this result is a new Szegő formula for certain pseudo–differential operators. The formula justifies the water–filling principle along time and frequency in terms of the time–continuous time–varying transfer function (the symbol).

I. INTRODUCTION

The information–theoretic treatment of the time–continuous channel dispersive in time and frequency (doubly–dispersive) with additive Gaussian noise has been a problem of long interest. A well known result for the time–invariant and power–limited case has been achieved by Gallager and Holsinger [1] and [2] in discretizing the time–continuous problem into an increasing sequence of parallel memoryless channels with known information capacity I_n . Coding theorems for the time–discrete Gaussian channel can be used for the time–continuous channel whenever such a discretization is realizable. A direct coding theorem without discretization has been established by Kadota and Wyner [3] for the causal, stationary and asymptotically memoryless channel.

The discretization in [2] was achieved by representing a single use of the time–continuous channel as the restriction of the channel operator to time intervals $\alpha\Omega$ of length α . The quantity I_n is then determined by spectral properties of the restricted operator. A major step in the calculation for the time–invariant case was the exact determination of the limit:

$$I(S) := \lim_{\alpha \rightarrow \infty} \left(\frac{1}{\alpha} \lim_{n \rightarrow \infty} I_n(\alpha S) \right) \quad (1)$$

which relies on the Kac–Murdock–Szegő result [4] on the asymptotic spectral behavior of convolution operators. As the classical result of Shannon for the time–continuous band–limited channel and the discussion in [5] shows, $I(S)$ has only a meaning of coding capacity for given power budget S whenever there exists a sequence of nested intervals of length α_k (i.e. realizable discretization) approaching this limit as $k \rightarrow \infty$. Some remaining problems in this direction, like for example the robustness of this limit against interference between different blocks, have been resolved for Gallager–Holsinger model in [6]. The limit has the advantage of nice interpretation as “water–filling” along the frequencies:

$$I(S) = \int_{B \cdot \sigma(\omega) \geq 1} \log(B \cdot \sigma(\omega)) d\omega \quad (2)$$

where σ denotes the symbol of the correlation operator L_σ (required to be absolute integrable and bounded). The constant

B is implicitly determined for a given power budget S by a relation similar to (2).

Since the time–invariant case represents the commutative setting a fixed signaling scheme (like for example orthogonal frequency division multiplexing) is permitted and the determination of the capacity is essentially reduced to a power allocation problem. Although the coherent setting (full knowledge at the transmitter) is considered so far only the channel gains have to be given to the transmitter in this case.

However, doubly–dispersive channels represent the non–commutative generalization and do not admit a joint diagonalization such that there still remains the problem of proper signal design. Here, the correlation operator can be characterized for example by the time–varying transfer function, i.e. the symbol $\sigma(x, \omega)$ of a so called pseudo–differential operator L_σ which depends on the frequency ω and the time instant x . Obviously, by uncertainty an exact characterization of frequencies at time instants is meaningless and the symbol can reflect spectral properties only in an averaged sense. Thus, it is important to know whether the limit in (1) for a real–valued symbol is asymptotically given by the average:

$$\frac{1}{\alpha} \iint_{\alpha\Omega \times \mathbb{R}} r(B \cdot \sigma(x, \omega)) dx d\omega \quad (3)$$

for $\alpha \rightarrow \infty$ and $r(x) = \log(x) \cdot \chi_{[1, \infty)}(x)$. Then, (3) with a similar integral with the function $(x - 1)/x \cdot \chi_{[1, \infty)}(x)$ represents the water–filling principle in time and frequency. Obviously, this strategy is used already in practice when optimizing rate functions in some long–term meaning. But, in fast–fading scenarios for example it not clear whether this procedure on a short time scale is indeed related to (1).

Averages closely related to the one in (3) have been studied for a long time in the context of asymptotic symbol calculus of pseudo–differential operators and semi–classical analysis in quantum physics [7], [8], [9]. Unfortunately, the results therein are not directly applicable in the information and communication theoretic setting because here 1.) the symbols of the restricted operators are (in general) discontinuous and usually not decaying in time 2.) the functions r to be considered are neither analytic nor have the required smoothness 3.) the path of approaching the limit has to be explicitly in terms of an increasing sequence of interval restrictions (infinite–dimensional subspaces) in order to establish its operational meaning. For operators with semigroup properties as for example the “heat channel” [10] it is possible to approach the limit via projections onto the (finite–dimensional) span of an increasing sequence of basis functions (Hermite functions in this case) as established in [11] for Schrödinger operators.

However, in the problem considered here this approach does not guarantee the existence of signaling schemes of finite length α to practically achieve the limit and a semigroup property of this particular type is not present.

The idea of approximate eigenfunctions of so called underspread channels [12], [13] has been used to obtain information-theoretical statements for the non-coherent setting [14]. Signal design has then to be considered with respect to statistical properties [15]. The method presented in this paper suggests that in the coherent setting the approximation in terms of trace norms is relevant.

A. Main Results

We establish a procedure for estimating the deviation of formula (3) from the desired quantity (1). It will be shown that both terms asymptotically agree for $\alpha \rightarrow \infty$ if the difference of symbol products $L_{\sigma\tau}$ and operator composition $L_{\sigma}L_{\tau}$ can be controlled in trace norm on $\alpha\Omega$ with a sub-linear scaling in α . We will further discuss the information-theoretical impacts: As an example we will study in more detail symbols $\sigma(x, \omega)$ which are Ω -periodic in x . We will show that under certain integrability and smoothness assumptions on the symbol the limit in (1) is indeed given as:

$$I(S) = \iint_{\Omega \times \mathbb{R}} r(B \cdot \sigma(x, \omega)) dx d\omega \quad (4)$$

whenever the (inverse) Fourier transform of $\sigma(x, \omega)$ in ω (the impulse response of L_{σ}) is supported in a fixed interval.

The paper is organized as follows: In Section II we introduce the channel model and establish the problem as a Szegő statement on the asymptotic symbol calculus for pseudo-differential operators. The asymptotic behavior is investigated in Section III as a series of four sub-problems: an increasing family of interval sections, the asymptotic symbol calculus, an approximation method and finally a result on "products" of symbols. Following this line of four arguments we are able to establish (4).

II. SYSTEM MODEL AND PROBLEM STATEMENT

We use $L_p(\Omega)$ for usual Lebesgue spaces ($1 \leq p \leq \infty$) of complex-valued functions on $\Omega \subseteq \mathbb{R}^n$ and abbreviate $L_p = L_p(\mathbb{R}^n)$ with corresponding norms $\|\cdot\|_{L_p}$. For $p = 2$ the Hilbert space has inner product $\langle u, v \rangle := \int \bar{u}v$. Classes of smooth functions up to order k are denoted with C^k and $\hat{f} = \mathcal{F}f$ is the Fourier transform of f . Partial derivatives of a function $\sigma(x, \omega)$ are written as σ_x and σ_{ω} , respectively. \mathcal{I}_2 and \mathcal{I}_1 are Hilbert-Schmidt and trace class operators with square-summable and absolute summable singular values and the symbol $\text{tr}(X)$ denotes the trace of an operator X (more details will be given later on) on L_2 .

A. System Model

We consider the common model of transmitting a finite energy signal s with support in an interval $\alpha\Omega$ of length α through a channel represented by a fixed linear operator H and additive distortion n_k , i.e. quantities measured at the receiver within some interval are expressed as noisy correlation responses:

$$\langle r_k, Hs \rangle + n_k \quad (5)$$

where $\{\langle r_k, \cdot \rangle\}$ are suitable normalized linear functionals implemented at the receiver. We assume Gaussian noise with $E(\bar{n}_k n_l) = \langle r_k, r_l \rangle$.

Let us denote with $(Pu)(x) = \chi(x/\alpha)u(x)$ the restriction of a function u onto the interval $\alpha\Omega$. Note that in what follows: P always depends on α . We will make in the following the assumption that the restriction HP of the channel operator H to input signals of length α with finite energy is compact, i.e. the restriction $PL_{\sigma}P$ of the correlation operator $L_{\sigma} := H^*H$ is compact as well (H^* denotes the adjoint operator on L_2). This excludes certain channel operators - like the identity - which are usually referred to as "dimension-unlimited", i.e. the wideband cases. Assume that the kernel $k(x, y)$ of L_{σ} fulfils for all $x \in \mathbb{R}$:

$$|k(x, x - z)|^2 \leq \psi(z) \quad (6)$$

for some $\sqrt{\psi} \in L_1 \cap L_2^1$. Then its (Kohn-Nirenberg) symbol or time-varying transfer function is given by Fourier transformation:

$$\sigma(x, \omega) = \int e^{i2\pi\omega(x-y)} k(x, x-y) dy \quad (7)$$

Throughout the paper we assume that σ is real-valued (this can be circumvented when passing to the Weyl symbol since L_{σ} is positive-definite). It follows that $\|\sigma(x, \cdot)\|_{L_2}^2 \leq \|\psi\|_{L_1}$ uniformly in x and that L_{σ} is bounded on L_2 :

$$\begin{aligned} |\langle u, L_{\sigma}v \rangle| &= |\langle u \otimes \bar{v}, k \rangle| \leq \langle u \otimes v, \sqrt{\psi} \rangle \\ &= \langle u, \sqrt{\psi} * v \rangle \leq \|\sqrt{\psi}\|_{L_1} \|u\|_{L_2} \|v\|_{L_2} \end{aligned} \quad (8)$$

From now on we use $\|\cdot\|_{\text{op}} := \|\cdot\|_{L_2 \rightarrow L_2}$ to denote the operator norm on L_2 . A compact operator HP can be written via the Schmidt representation (singular value decomposition) as a limit of a sum of rank-one operators $HP = \sum_k s_k \langle u_k, \cdot \rangle v_k$ with singular values $s_k = \sqrt{\lambda_k(PL_{\sigma}P)}$ and orthonormal bases $\{u_k\}$ and $\{v_k\}$ - all depending on α . For the coherent setting we assume that finite subsets of these bases are known and implementable at the transmitter and the receiver, respectively. Obviously, this is an idealized and seriously strong assumption which can certainly not be fulfilled without error in practise. The investigations in [16] suggest that underspreadness of H is necessary prerequisite for reliable error control. When representing the signal s as a finite linear combination of $\{u_k\}$ a single use of the time-continuous channel H over the time interval $\alpha\Omega$ with power budget S is decomposed into a single use of a finite set of time-discrete parallel Gaussian channels jointly constrained to αS .

We will consider in the following independent uses of the channel in (5) as our preliminary² model and restrict to $r_k = v_k$, i.e. $E(\bar{n}_k n_l) = \delta_{kl}$. Then, the capacity and the power budget of the equivalent memoryless Gaussian channel are related through the water-filling level B as (see for example [2]):

$$\begin{aligned} \frac{1}{\alpha} \sum_{B\lambda_k \geq 1} \log(B\lambda_k) &= \frac{1}{\alpha} \text{tr}_{\alpha} r(B PL_{\sigma}P) \\ \frac{B}{\alpha} \sum_{B\lambda_k \geq 1} \frac{B\lambda_k - 1}{B\lambda_k} &= \frac{B}{\alpha} \text{tr}_{\alpha} p(B PL_{\sigma}P) \end{aligned} \quad (9)$$

¹ $\sup_{x \in \mathbb{R}} k(x, x - \cdot) \in L_1 \cap L_2$

²We discuss consecutive uses of the same time-continuous channel below.

with $r(x) = \log(x) \cdot \chi_{[1,\infty)}(x)$ and $p(x) = \frac{x-1}{x} \cdot \chi_{[1,\infty)}(x)$. The symbol $\text{tr}_\alpha Y := \text{tr}(PY P)$ denotes the trace of the operator Y on the range of P and the operators $r(PXP)$ and $p(PXP)$ for X being self-adjoint are meant by the spectral mapping theorem.

If the time-varying impulse response of L_σ (or H) has finite delay ($k(x, x-z)$ is zero for z outside a fixed interval) and is periodic in the time instants x (the symbol $\sigma(x, \omega)$ is periodic in x) multiple channel uses in the preliminary model can be taken as consecutive uses of the same time-continuous channel. Inserting guard periods of appropriate fixed size (independent of α) will not affect the asymptotic behavior for $\alpha \rightarrow \infty$. Thus, any further results will then indeed refer to the information (and coding) capacity. The assumptions on finite delay might be relaxed using direct methods like in [6] or [17] whereby extensions to almost-periodic channels seems to lie at the heart of information theory.

B. Problem Statement

The interval restriction P has the symbol $\chi(x/\alpha)$. The symbol of operator products is given as the twisted multiplication of the symbol of the factors. Under the trace this is reduced to ordinary multiplication (see for example [18] in the case of Weyl correspondence). Thus, the term in (3) can be written as the following trace:

$$\frac{1}{\alpha} \text{tr}_\alpha L_{f(\sigma)} = \frac{1}{\alpha} \int_{\alpha\Omega \times \mathbb{R}} f(\sigma(x, \omega)) dx d\omega \quad (10)$$

when taking $f(x) = r(Bx)$. Comparing (9) with (10) means to estimate the asymptotic behavior of:

$$\frac{1}{\alpha} \text{tr}_\alpha (f(PL_\sigma P) - L_{f(\sigma)}) \quad (11)$$

for $\alpha \rightarrow \infty$ (we abbreviate $f(\sigma) := f \circ \sigma$). As seen from r and p in (9) the functions f of interest are continuous but not differentiable at $x = 1$.

III. ASYMPTOTIC TRACE FORMULAS

The procedure for estimating the difference in (11) essentially consists in the following arguments: A functional calculus will be used to represent the function f in the operator context. For $L_{f(\sigma)}$ this can be done independently of α but for $f(PL_\sigma P)$ such an approach is much more complicated because of the remaining projections P . Hence, the first step is to estimate its deviation to $f(L_\sigma)$ by inserting the zero term $\text{tr}_\alpha (f(L_\sigma) - f(L_\sigma))/\alpha$ into (11):

$$\frac{1}{\alpha} \left(\overbrace{\text{tr}_\alpha [f(PL_\sigma P) - f(L_\sigma)]}^{\text{stability}} + \overbrace{\text{tr}_\alpha [f(L_\sigma) - L_{f(\sigma)}]}^{\text{symbol calculus}} \right) \quad (12)$$

and use $|\text{tr}(a+b)| \leq |\text{tr} a| + |\text{tr} b|$ to estimate both terms separately. The first contribution refers to the stability of interval sections (in Section III-A). For second term a Fourier-based functional calculus reduces the problem to the characterization of the approximate product rule for symbols (in Section III-B) which can then be estimated independently of the particular function f (in Section III-D). Unfortunately, the last steps require certain smoothness of f . Therefore we will approach the limit via smooth approximations f_ϵ as discussed in Section III-C.

A. Stability of Interval Sections

The following stability result was inspired by the analysis on the Widom conjecture in [19]. Let $\text{spec}(L_\sigma)$ denote the spectrum of L_σ . Then the interval $I := \bigcup_{t \in [0,1]} t \cdot \text{spec}(L_\sigma)$ contains the spectra of the family $PL_\sigma P$ for each α .

Theorem 1. *Let L_σ be an operator with a kernel which fulfils $|k(x, x-z)|^2 \leq \psi(z)$ with $\psi \in L_1$. If $\|\psi(1 - \chi_{[-s,s]})\|_{L_1} \leq c/s$ then:*

$$\frac{1}{\alpha} |\text{tr}_\alpha (f(PL_\sigma P) - f(L_\sigma))| \leq \|f''\|_{L_\infty(I)} \frac{\log(\alpha)}{\alpha} \quad (13)$$

for $f \in W_\infty^2(I)$.

$W_\infty^2(I)$ denotes the Sobolev class (details in [20]). Recall that the functions f to be considered here are continuous and differentiable a.e. on I (except at point $x = 1$). We will shortly discuss the proof of this theorem since it is only a minor variation of [19].

Proof: Laptev and Safarov [20] have obtained from Berezin inequality the following estimate. For functions $f \in W_\infty^2(I)$ the operator $P[f(L_\sigma) - f(PL_\sigma P)]P$ is trace class if PL_σ and $PL_\sigma(1 - P)$ are Hilbert-Schmidt with the trace estimate:

$$|\text{tr}_\alpha (f(L_\sigma) - f(PL_\sigma P))| \leq \frac{1}{2} \|f''\|_{L_\infty(I)} \|PL_\sigma(1 - P)\|_{\mathcal{L}_2}^2 \quad (14)$$

Recall that the interval projection P is multiplication with the scaled characteristic function $\chi(x/\alpha)$. Thus, change of variables $x = y' + x'$ and $y = y' - x'$ gives:

$$\begin{aligned} \|PL_\sigma\|_{\mathcal{L}_2}^2 &= \int \chi(x/\alpha) |k(x, y)|^2 dx dy \\ &\leq 2\alpha^2 \int \psi(2\alpha x') dx' \int \chi(y' + x') dy' \leq \alpha \|\psi\|_1 \end{aligned} \quad (15)$$

In the same manner we get:

$$\begin{aligned} \|PL_\sigma(1 - P)\|_{\mathcal{L}_2}^2 &= \int \chi\left(\frac{x}{\alpha}\right) (1 - \chi\left(\frac{y}{\alpha}\right)) |k(x, y)|^2 dx dy \\ &\leq \alpha^2 \int \chi(x) (1 - \chi(y)) \psi(\alpha(x - y)) dx dy \\ &= \alpha^2 \int \psi(2\alpha x) \cdot \omega(2x) dx \end{aligned} \quad (16)$$

with $\omega(x) := 4|x| \leq 2$ for $|x| \leq 1/2$ and $\omega(x) := 2$ outside this interval. With $u = 2\alpha x$ and $\phi(u) = \psi(u) + \psi(-u)$ we split and estimate the integral as follows:

$$\begin{aligned} \|PL_\sigma(1 - P)\|_{\mathcal{L}_2}^2 &= \frac{\alpha}{2} \int_0^\infty \phi(u) \omega\left(\frac{u}{\alpha}\right) du \\ &\leq \frac{\alpha}{2} \left(\frac{8}{\alpha} \int_0^2 + \int_2^{2\alpha} \frac{4u}{\alpha} + 2 \int_{2\alpha}^\infty \right) \phi(u) du \end{aligned} \quad (17)$$

and with the assumptions of the theorem it follows:

$$\|PL_\sigma(1 - P)\|_{\mathcal{L}_2}^2 = 4\|\psi\|_1 + 2 \int_2^{2\alpha} \phi(u) u du + \frac{c}{2} \quad (18)$$

Finally we use $\phi(u) = -\frac{d}{du} \int_u^\infty \phi(s) ds$ and integrate by parts to obtain $\int_2^{2\alpha} \phi(u) u du = c(1 + \log \alpha)$. ■

B. Asymptotic Symbol Calculus

Here we shall use Fourier techniques to estimate the right term in (12). We abbreviate in the following $e(x) = \exp(i2\pi x)$.

Lemma 2. *Let f be a L_1 -function with $\hat{f}(\omega) = \mathcal{O}(\omega^{-4-\delta})$ for some $\delta > 0$. For L_σ being bounded and self-adjoint on L_2 with real-valued symbol $\sigma \in C^3$ it follows that:*

$$\frac{1}{\alpha} |\text{tr}_\alpha (f(L_\sigma) - L_{f(\sigma)})| \leq \int dw |\hat{f}(\omega)| \int_0^\omega Q_\alpha(s) \frac{ds}{\alpha} \quad (19)$$

with $Q_\alpha(s) := \|(L_\sigma L_{e(s\sigma)} - L_{\sigma e(s\sigma)}) P\|_{\mathcal{I}_1}$.

The lemma shows that whenever the rhs in (19) is finite the asymptotics for $\alpha \rightarrow \infty$ is determined only by Q_α/α . The function Q_α essentially compares the twisted product of σ and $e(s\sigma)$ with the ordinary product $\sigma \cdot e(s\sigma)$ in trace norm reduced to intervals of length α .

Proof: Consider the following operator-valued Bochner integral:

$$f(L_\sigma) = \int e(\omega L_\sigma) \hat{f}(\omega) d\omega \quad (20)$$

where the operator $e(\omega L_\sigma)$ is defined as the usual power series converging in norm since L_σ is bounded. In particular $e(\omega L_\sigma)$ is unitary on L_2 (L_σ is self-adjoint) and depends continuously on ω . Since $\|f(L_\sigma)\|_{\text{op}} \leq \|\hat{f}\|_1$ convergence in operator norm is guaranteed and the construction agrees with the spectral mapping theorem (see [21]). The value of the symbol $f \circ \sigma$ at each point can be expressed in terms of \hat{f} . This suggests the formula:

$$L_{f(\sigma)} = \int L_{e(\omega\sigma)} \hat{f}(\omega) d\omega \quad (21)$$

From Calderon Vaillancourt Theorem [22, Ch.5] we have:

$$\|L_{e(s\sigma)}\|_{\text{op}} \leq \|e(s\sigma)\|_{C^3} := \sum_{a+b \leq 3} |2\pi s|^{a+b} \|\partial_x^a \partial_\omega^b \sigma\|_{L_\infty} \quad (22)$$

Thus, for $\hat{f}(\omega) = \mathcal{O}(\omega^{-4-\delta})$ and $\delta > 0$ the integral (21) converge in the sense of Bochner. From the considerations above we get therefore:

$$|\text{tr}_\alpha (L_{f(\sigma)} - f(L_\sigma))| \leq \int |\hat{f}(\omega)| \cdot |\text{tr}_\alpha u(\omega)| d\omega \quad (23)$$

with $u(\omega) = L_\sigma e(\omega L_\sigma) - L_{\sigma e(\omega\sigma)}$. As suggested in [9] the operator $u(\omega)$ fulfils the following identity³:

$$u'(\omega) = i2\pi (L_\sigma u(\omega) + L_\sigma L_{e(\omega\sigma)} - L_{\sigma e(\omega\sigma)}) \quad (24)$$

i.e. an inhomogenous Cauchy problem with initial condition $u(0) = 0$. By Duhamel's principle (see for example [24, p.50] for the Banach-space valued case):

$$u(\omega) = \frac{2\pi}{i} \int_0^\omega e((\omega - s)L_\sigma) (L_\sigma L_{e(s\sigma)} - L_{\sigma e(s\sigma)}) ds \quad (25)$$

giving the estimate:

$$|\text{tr}_\alpha u(\omega)| \leq \int_0^\omega \|(L_\sigma L_{e(s\sigma)} - L_{\sigma e(s\sigma)}) P\|_{\mathcal{I}_1} ds \quad (26)$$

since $\|Pe((t-s)L_\sigma)\|_{\text{op}} \leq 1$. ■

³in case of operators: $\partial_\omega e(\omega L_\sigma) = i2\pi L_\sigma e(\omega L_\sigma)$ [23, Lemma 5.1].

The smoothness assumptions in the theorem can be weakened to $\sigma \in C^{2+\delta}$ and $\hat{f}(\omega) = \mathcal{O}(\omega^{-3-\delta})$ when using Hölder-Zygmund spaces. We expect that these conditions can be further reduced when using in (21) some weaker convergence in tr_α instead of requiring a Bochner integral. The proof of the theorem can also be based on the Paley-Wiener theorem, i.e. $f \rightarrow f(L_\sigma)$ and $f \rightarrow L_{f(\sigma)}$ are operator-valued distributions of compact support with order at most 3 and have therefore C^3 as natural domain.

C. An Approximation Procedure

Since L_σ is bounded (see (8)) the functions f will be evaluated only on a finite interval contained in I . We consider functions f of the form $f(x) = h(x) \cdot \chi_{[1,\infty)}(x)$ with a critical point at $x = 1$. By smooth extension outside the interval its Fourier transforms $\hat{f}(\omega)$ decay only as $\mathcal{O}(\omega^{-2})$, see here for example [25, Theorem 2.4], i.e. $f \in L_1 \cap \mathcal{FL}_1$. Unfortunately, this is not sufficient for Lemma 2. Therefore, we replace the Heaviside function $\chi_{[1,\infty)}$ in f by a series of smooth approximations ϕ_ϵ as done for example in [8]. Let be $\phi \in C^\infty$ with $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = 1$ for $t \geq 1$. Define $\phi_\epsilon(x) = \phi(\frac{x-1}{\epsilon})$ and consider $f_\epsilon = h\phi_\epsilon \in C_c^\infty$ (a smooth function of compact support, achieved again by smooth extension outside the interval I) instead of f :

$$|\hat{f}_\epsilon(\omega)| \leq \frac{c'_n |I|}{|2\pi\omega|^n} \epsilon^{-n} \quad (27)$$

In essence: polynomial grow of $Q_\alpha(s)$ in s can always be compensated by taking n large enough such that at the rhs in (19) remains a finite quantity $R_\alpha(\epsilon)$. If for example $R_\alpha(\epsilon) = \mathcal{O}(\alpha^{-\gamma})$, we choose $\epsilon = \alpha^{-\delta}$ with $\delta < \gamma/n$. Then $R_\alpha(\epsilon) \rightarrow 0$ and $\epsilon \rightarrow 0$ for $\alpha \rightarrow \infty$ which is obviously sufficient for the limit.

D. Approximate Symbol Products

Let us abbreviate $\tau = e(s\sigma)$. Then the operator in the term $Q_\alpha(s)/\alpha$ of Lemma 2 is the deviation between operator and symbol product $L_\sigma L_{\sigma\tau} - L_{\sigma\tau}$. As in [7] we insert $L_\sigma L_\tau^* - L_\sigma L_\tau^*$ and apply triangle inequality to obtain:

$$Q_\alpha(s) \leq \|L_\sigma\|_{\text{op}} \|TP\|_{\mathcal{I}_1} + \|T'P\|_{\mathcal{I}_1} \quad (28)$$

where $T = L_\tau^* - L_\tau$ and $T' = L_\sigma L_\tau^* - L_{\sigma\tau}$ having kernels $t(x, y)$ and $t'(x, y)$:

$$\begin{aligned} t(x, y) &= \int e^{i2\pi(x-y)\omega} (\tau(x, \omega) - \tau(y, \omega)) d\omega \\ t'(x, y) &= \int e^{i2\pi(x-y)\omega} \sigma(x, \omega) (\tau(x, \omega) - \tau(y, \omega)) d\omega \end{aligned} \quad (29)$$

Polynomial orders in s which will occur in the following will be compensated by the approximation method in Section III-C. The role of τ and σ can also be interchanged since according (22) L_τ is bounded polynomially in s .

We will discuss in the following under which conditions $\|TP\|_{\mathcal{I}_1}$ is finite and what will be scaling in α . The argumentation for $\|T'P\|_{\mathcal{I}_1}$ will be analogous. From integration by parts (since $\tau(x, \omega) - \tau(y, \omega) \rightarrow 0$ for $|\omega| \rightarrow \infty$) we have:

$$t(x, y) = \frac{h(x, x-y) - h(y, x-y)}{i2\pi(x-y)} \quad (30)$$

where $h(x, z) = \int e^{i2\pi\omega z} \tau_\omega(x, \omega) d\omega$ and $|\tau_\omega| = |2\pi s \sigma_\omega|$. It is already assumed that $\sigma(x, \cdot) \in L_2$ uniformly in x . If $\sigma_\omega(x, \cdot)$ is of bounded variation (or even continuous) we deduce with the mean value theorem that $|t(x, y)|^2 \leq c/(1 + |x - y|^2)$. This in turns implies that $\|TP\|_{\mathcal{I}_2} = \mathcal{O}(\sqrt{\alpha})$ but it will not be sufficient for $\|TP\|_{\mathcal{I}_1}$.

It is known that general trace class estimates can not be achieved in this way and further smoothness assumptions are necessary. The problem is related to the absolute summability of orthogonal series (in particular Fourier series as shown later on) which is evident from:

$$\|TP\|_{\mathcal{I}_1} \leq \sum_n \|T\phi_n\|_{L_2} \quad (31)$$

where ϕ_n is an ONB for the range of P ($\text{supp } \phi_n \subseteq \alpha\Omega$). However, for finite-rank TP it follows here already from $\|T\phi_n\|_{L_2} \leq \|TP\|_{\mathcal{I}_2} = \mathcal{O}(\sqrt{\alpha})$ that the approximation method in Section III-C can be applied giving the correct statement in (11).

Let be c_n a (positive) sequence $1/c_n \rightarrow 0$ as $n \rightarrow \infty$ with $K = \sum_n c_n^{-2\lambda}$ being finite for $\lambda > 1/2$. Hölder inequality implies:

$$\begin{aligned} \|TP\|_{\mathcal{I}_1}^2 &\leq K \sum_n \|c_n^\lambda T\phi_n\|_{L_2}^2 \\ &= K \int_{\mathbb{R}} \sum_n |\langle \bar{t}(z + \cdot, \cdot), c_n^\lambda \phi_n \rangle|^2 dz \end{aligned} \quad (32)$$

The periodic case: Let us assume that $t(z + y, y)$ is periodic in y (same for t') with period 1 (for simplicity) which is given if the symbol $\sigma(x, \omega)$ is 1-periodic in x . We use the Fourier basis $\phi_n(y) = \exp(i2\pi ny/\alpha)/\sqrt{\alpha}$ and consider $\alpha \in \mathbb{N}$ and $\Omega = [0, 1]$, i.e. t is given by the series:

$$t(z + y, y) = \sqrt{\alpha} \sum_m \hat{t}_m(z) \phi_{\alpha m}(y) \quad (33)$$

The sum in (32) reduces to the indexes αn . We take exemplary $\lambda = 1$, $c_{\alpha n} = 2\pi n$ and $c_{\alpha n+k} = 2\pi\sqrt{\alpha-1}n$ for $k = 1 \dots \alpha-1$ such that K is independent of α . Then (32) is:

$$\|TP\|_{\mathcal{I}_1} \leq \sqrt{\alpha} \left(K \int \sum_m |2\pi m \hat{t}_m(z)|^2 dz \right)^{1/2} \quad (34)$$

which results in the condition $t_y \in L_2(\mathbb{R} \times \Omega)$ (to be precise, the derivative in the L_2 -mean - see [26]). From the definition of t in (29) it follows that:

$$t_y(z + y, y) = \int e^{i2\pi z \omega} (\tau_x(y + z, \omega) - \tau_x(y, \omega)) d\omega \quad (35)$$

where $|\tau_x| = 2\pi s |\sigma_x|$. Thus, the latter condition is fulfilled if $\sigma_x(x, \cdot) \in L_1$ and the derivative in ω has bounded variation (or is continuous). Thus, in this case the method in Section III-C can be applied which proves formula (4).

IV. CONCLUSION

A new approach to the capacity of time-continuous doubly-dispersive Gaussian channels with periodic symbol has been established by proving an asymptotic Szegő result for certain pseudo-differential operators.

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